The Finite-Element Formulation to Plane Flow Problems

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Outline of presentation

- Introduction to metal formulation
  - Yield Criterion
  - Equilibrium
  - Plastic potential and flow rule.

- Methods of Analysis

- The Finite Element Method
  - Basis for the Finite–Element Formulations
  - Finite–Element Procedures

- Plane–Strain Problems

- References
Plasticity

Basic quantities used to describe the mechanics of deformation in plastically deforming solids:

1. Stress
2. Strain–rate

These are expressed with respect to a fixed coordinate system in the material configuration at a time under consideration.

Two models: Lagrangian – particle in the reference state as the independent variable.

Eulerian – point in the deformed state as the independent variable.

Infinitesimal deformation theory $E \approx e$
1. Flow formulation, based on infinitesimal deformations of a continuum body – use of the Cauchy's strain tensor, $\varepsilon$

$$\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i})$$

2. Solid formulation, based on finit deformation.
Strain–rate components are time derivatives of strain components

\[
\dot{\varepsilon}_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})
\]

This time \( u_{ij} \), represents the time derivative of displacements, thus, velocities.

\[
\dot{\gamma}_{ij} = \text{Engineering strain–rate components}
\]

\[
\begin{align*}
\dot{\varepsilon}_x &= \frac{\partial u_x}{\partial x}, \\
\dot{\varepsilon}_y &= \frac{\partial u_y}{\partial y}, \\
\dot{\varepsilon}_z &= \frac{\partial u_z}{\partial z}
\end{align*}
\]

\[
\begin{align*}
\dot{\varepsilon}_{xy} &= \frac{1}{2} \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) = \frac{\dot{\gamma}_{xy}}{2} \\
\dot{\varepsilon}_{yz} &= \frac{1}{2} \left( \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \right) = \frac{\dot{\gamma}_{yz}}{2} \\
\dot{\varepsilon}_{zx} &= \frac{1}{2} \left( \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z} \right) = \frac{\dot{\gamma}_{zx}}{2}
\end{align*}
\]
Cauchy stress tensor

It may be specified by the three principal components – invariants. 

\[
[\sigma_{ij}] = \begin{bmatrix}
\sigma_{11} & \sigma_{21} & \sigma_{31} \\
\sigma_{12} & \sigma_{22} & \sigma_{32} \\
\sigma_{13} & \sigma_{23} & \sigma_{33}
\end{bmatrix} = \begin{bmatrix}
\sigma_x & \tau_{yx} & \tau_{zx} \\
\tau_{xy} & \sigma_y & \tau_{zy} \\
\tau_{xz} & \tau_{yz} & \sigma_z
\end{bmatrix}
\]

These are the roots of the cubic equation

\[
\sigma^3 - I_1 \sigma^2 - I_2 \sigma - I_3 = 0
\]

\(I_1, I_2, I_3\) are defined by the follow relationships:

\[
I_1 = \sigma_1 + \sigma_2 + \sigma_3 \\
I_2 = -\left(\sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_1\right) \\
I_3 = \sigma_1 \sigma_2 \sigma_3
\]
The stress tensor can be expressed as a sum of two other stress tensors: deviatoric stress tensor $\sigma_{ij}'$ and hydrostatic stress $\sigma_m = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$.

\[ \sigma_{ij} = \sigma_{ij}' + \delta_{ij}\sigma_m \]

Since

\[ J_1 = \sigma_1' + \sigma_2' + \sigma_3' = 0 \]

we use only two invariants

\[ J_2 = -(\sigma_1'\sigma_2' + \sigma_2'\sigma_3' + \sigma_3'\sigma_1') \]
\[ J_3 = \sigma_1'\sigma_2'\sigma_3' \]
The Yield Criteria

Yield Function

\[ f(\sigma_{ij}) = C \text{(constant)} \]

Isotropic materials, plastic yielding can depend only in the magnitude of three principal stresses and not their directions.

\[ f(I_1, I_2, I_3) = C \]

Experimentally and in a first approximation, yielding of a material is unaffected by a moderate hydrostatic stress. It can be used the invariants of the deviatoric stress. This way,

\[ f(J_2, J_3) = C \]
Two criteria
1. Tresca’s criteria
   \[ \sigma_1 - \sigma_3 = \text{const.} \]
2. von Mises Criterion

\[ f(\sigma_{ij}) = J_2 \]

\( k \) depends on the material properties

The constants can be determined from simple states as in uniaxial tension (\( \sigma_1 = Y ; \sigma_2 = \sigma_3 = 0 \))

\[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2Y^2 \]

\[ k = \frac{Y}{\sqrt{3}} \]
Equilibrium equations, if the body force is neglected:

\[
\sigma_{ij,i} = 0
\]

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0
\]

\[
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = 0
\]

\[
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0
\]

The stress along the boundary surface is in equilibrium with an applied traction

\[
F_i = \sigma_{ij} n_j
\]

\(n_j\) outward normal
Plastic Potential and Flow rule

Stress and plastic strain relationship

Plastic strain–rate:

\[ \dot{\varepsilon}_{ij}^p = h \frac{\partial g}{\partial \sigma_{ij}} \dot{f} \]

\( g \) and \( h \) – scalar functions of invariants of the deviatoric stresses

\( f \) – yield function

If the flow function is associative, plastic potential \( g(\sigma_{ij}) \) is equivalent to the yield function

\[ \dot{\varepsilon}_{ij}^p = \frac{\partial f}{\partial \sigma_{ij}} \dot{\lambda} \]

\( \dot{\lambda} = hf \) is a positive proportionality constant
Flow stress

It is defined as the instantaneous value of stress required to continue deforming the material – to keep the metal flowing

\[
\bar{\sigma} = \sqrt{\frac{1}{2}} \{ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \}^{1/2} = \sqrt{3} \ J_2
\]

Plastic work per unit volume during a certain finite deformation

\[
W_p = \int \sigma_{ij} \, d\varepsilon_{ij} \quad dW_p = \sigma_{ij} \, d\varepsilon_{ij} = \bar{\sigma} \, d\ddot{\varepsilon}
\]

Work–rate using flow rule

\[
\dot{W}_p = \sigma_{ij} \frac{\partial f}{\partial \sigma_{ij}} \dot{\lambda} = 2f(\sigma_{ij}) \dot{\lambda} = \bar{\sigma} \dot{\varepsilon}
\]

\[
\dot{\lambda} = \frac{3 \dot{\varepsilon}}{2 \bar{\sigma}}
\]

\[
\dot{\varepsilon} = \sqrt{\frac{2}{3}} \{ \dot{\varepsilon}_{ij} \dot{\varepsilon}_{ij} \}^{1/2}
\]
When it comes to analyzing metal-forming processes, the governing equations for the solution of the mechanics of plastic deformation of rigid-plastic and rigid-viscoplastic materials are summarized as follows:

**Equilibrium equations:** \[ \frac{\partial \sigma_{ij}}{\partial x_j} = 0 \]

**Yield criterion,** \( f(\sigma_{ij}) = C: \) \[ \bar{\sigma} = \sqrt{\frac{3}{2}} (\sigma_{ij}' \sigma_{ij}')^{1/2} = \bar{\sigma}(\bar{\epsilon}, \dot{\bar{\epsilon}}) \]

**Constitutive equations:** \[ \dot{\epsilon}_{ij} = \frac{\partial f(\sigma_{ij})}{\partial \sigma_{ij}} \dot{\lambda} \]

\[ \dot{\epsilon}_{ij} = \frac{3}{2 \bar{\sigma}} \sigma_{ij}' \]

with \( \dot{\epsilon} = \sqrt{\frac{2}{3}} (\dot{\epsilon}_{ij} \dot{\epsilon}_{ij})^{1/2} \)

**Compatibility conditions:** \[ \dot{\epsilon}_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \]
• The unknowns for the solution of a quasi-static plastic deformation process are 6 stress components and 3 velocity components;

• The governing equations are:
  • 3 equilibrium equations;
  • the yield condition;
  • 5 strain-rate ratios derived from the flow rule.

• The boundary conditions are prescribed in terms of velocity and traction. Along the tool-workpiece interface, the velocity component is prescribed in the direction normal to the interface and the traction is specified by the frictional stress in the tangential direction.

Since it is difficult to obtain a complete solution that satisfies all of the governing equations, various approximate methods have been devised, depending upon the assumptions and approximations.
The basic principles and concepts involved in the finite-element method are the variational principle and discretization. In the course of developing the analysis methods, the variational principle has played a significant part in expanding analysis capabilities to prediction of phenomena of industrial importance. These methods are specifically:

- the upper-bound method;
- Hill’s general method,

leading to the finite element method.

In the finite-element approach, because of the discrete representation of approximating velocity fields, the class of considered velocity fields is much wider than the ones we get from upper-bound and Hill’s methods.
The Finite Element Method

There are four approaches for the derivation of the basic equations for the finite-element analysis:

- The direct approach;
- The variational method;
- The method of weighted residuals;
- The energy balance approach.

In the following, the variational method and a special case of the weighted residual method are used \([1]\) to obtain the basic equation for the finite-element discretization. Once the solution for the velocity field that satisfies the basic equation is obtained, then corresponding stresses can be calculated using the flow rule and the known mean stress distribution.
Basis for the Finite-Element Formulation

The variational approach is based on one of two variational principles. It requires that among admissible velocities $u_i$ that satisfy the conditions of compatibility and incompressibility, as well as the velocity boundary conditions, the actual solution gives the following functional a stationary value:

$$\pi = \int_V \bar{\sigma} \dot{\epsilon} \, dV - \int_{S_F} F_i u_i \, dS,$$

for rigid-plastic materials \hspace{1cm} (a)

and

$$\pi = \int_V E(\dot{\varepsilon}_{ij}) \, dV - \int_{S_F} F_i u_i \, dS,$$

for rigid-viscoplastic materials

Where $\bar{\sigma}$ is the effective stress, $\dot{\epsilon}$ is the effective strain-rate, $F_i$ represents surface tractions, and $E(\dot{\varepsilon}_{ij})$ is the work function.
The following equations are obtained either:

- using the solution of the original boundary–value problem, obtained from the solution of the dual variational problem (where the first–order variation of the functional vanishes) \(^1\), or

- beginning with a weak form of the equilibrium equations \(^1\)

\[
\delta \pi = \int_V \bar{\sigma} \delta \dot{\varepsilon} \, dV + \int_V \lambda \delta \dot{\varepsilon}_v \, dV + \int_V \dot{\varepsilon}_v \delta \lambda \, dV - \int_{S_F} F_i \delta u_i \, dS = 0 \tag{b}
\]

\[
\delta \pi = \int_V \bar{\sigma} \delta \dot{\varepsilon} \, dV + K \int_V \dot{\varepsilon}_v \delta \dot{\varepsilon}_v \, dV - \int_{S_F} F_i \delta u_i \, dS = 0 \tag{c}
\]

Where \(\lambda\) is a Lagrange multiplier, \(\dot{\varepsilon}_v = \dot{\varepsilon}_{ii}\) is the volumetric strain–rate, \(K\) is a penalty (very large) constant, \(\delta u_i\) and \(\delta \lambda\) are arbitrary variations and \(\delta \dot{\varepsilon}\) and \(\delta \dot{\varepsilon}_v\) are the variations in strain–rate derived from \(\delta u_i\). The first or the second equations are the basic equation for the finite–element formulation.
Discretization of a problem consists of the following steps:

- describing the element;
- setting up the element equation;
- assembling the element equations.

Numerical analysis techniques are then applied for obtaining the solution of the global equations, using the equations (a) and (b) or (c).

The solution satisfying the equation (a) is obtained from the admissible velocity fields that are constructed by introducing the shape function in such a way that a continuous velocity field over each element can be defined uniquely in terms of velocities of associated nodal points.
In the deformation process shown, the workpiece is divided into elements, without gaps or overlaps between elements.

In order to ensure continuity of the velocities over the whole workpiece, the shape function is defined such that the velocities along any shared element–side are expressed in terms of velocity values at the same shared set of nodes (compatibility requirement).

Then a continuous velocity field over the whole workpiece can be uniquely defined in terms of velocity values at nodal points specified globally.

\[ \mathbf{v}^T = \{v_1, v_2, \ldots, v_N\} \]

Set of nodal points velocities in a vector form, where 
\[ N = (\text{total nº of nodes}) \times (\text{degrees of freedom per node}) \].
An admissible requirement for the velocity field is that the velocity boundary condition prescribed on the surface $S_u$ (essential boundary condition) must be satisfied.

This condition can be imposed at nodes on $S_u$ by assigning known values to the corresponding variables.

- The incompressibility condition is not required for defining a velocity field in formulation of eq. (b) or (c).

Equations (a) and (b) or (c) are now expressed in terms of nodal point velocities $v$ and their variations $\delta v$. From arbitrariness of $\delta v_i$, a set of algebraic equations (stiffness equations) are obtained as

$$\frac{\partial \pi}{\partial v_i} = \sum_j \left( \frac{\partial \pi}{\partial v_i} \right)_{(j)} = 0 \quad \text{(d)}$$

where $(j)$ indicates the quantity at the $j$th element. The capital letter suffix refers to the nodal point number.

This equation is obtained by evaluating the $(\partial \pi/\partial v_i)$ at the elemental level and assembling them into the global equation under appropriate constrains.
In metal-forming, the stiffness equation (d) is nonlinear and the solution is obtained iteratively by using the Newton–Raphson method, which consists of linearization and application of convergence criteria to obtain the final solution.

Linearization is achieved by Taylor expansion near an assumed solution point \( \mathbf{v} = \mathbf{v}_0 \) (initial guess), namely,

\[
\left[ \frac{\partial \pi}{\partial v_I} \right]_{v=v_0} + \left[ \frac{\partial^2 \pi}{\partial v_I \partial v_J} \right]_{v=v_0} \Delta v_J = 0 \quad (e)
\]

where \( \Delta v_J \) is the first-order correction of the velocity \( \mathbf{v}_0 \).

Equation (e) can be written in the form: \( \mathbf{K} \Delta \mathbf{v} = \mathbf{f} \) (f), where \( \mathbf{K} \) is called the stiffness matrix and \( \mathbf{f} \) is the residual of the nodal point force vector.

Once the solution of eq. (f) for the velocity term \( \Delta \mathbf{v} \) is obtained, the assumed velocity \( \mathbf{v}_0 \) is updated according to \( \mathbf{v}_0 + \alpha \Delta \mathbf{v} \), where \( \alpha \) is a constant between 0 and 1 called the deceleration coefficient. Iteration is continued until the velocity correction terms become negligibly small (with an error norm close to \( 5 \times 10^{-5} \)).
The Newton–Raphson iteration process is shown schematically in the Fig. It is seen from the figure that convergence of this process iterations depends on the initial guess velocity, it has to be close to the actual solution.

When a deformation process is relatively simple, the initial guess velocity can be provided, for instance, for the upper bound method. However, if the process is complex and obtaining a good initial guess solution is difficult, the use of other methods, such as direct iteration method, may be more appropriated.

Two convergence criteria may be used:
- one measures the error norm of the velocities;
- the other requires the norm of residual equations.
The finite-element method procedures outlined above are implemented in a computer program in the following way:

1. Generate an assumed velocity;
2. Evaluate the elemental stiffness matrix for the velocity correction term \( \Delta \mathbf{v} \) in eq. (f);
3. Impose velocity conditions to the elemental stiffness matrix, and repeat step 2 over all elements defined in the workpiece;
4. Assemble elemental stiffness matrix to form a global stiffness equation;
5. Obtain the velocity correction terms by solving the global stiffness equation;
6. Update the assumed velocity. Repeat steps 2 through 6 until the velocity solution converges;
7. When the converged velocity solution is obtained, update the geometry of the workpiece using the velocity nodes during a time increment. Step 2 through 7 are repeated until the desired degree of deformation is achieved.

The above procedure applies to the analysis of nonsteady-state processes. For steady-state processes, updating the geometry of the workpiece is not necessary.
Formulations and solutions for plane plastic flow.

Velocities in all points occur in planes parallel to a certain plane and are independent of the distance to that plane.

For example, plane \((x,y)\). The Cartesian components of the velocity vector \(u\) are:

\[
\begin{align*}
u_x(x,y) & \quad u_y(x,y) & \quad u_z = 0
\end{align*}
\]
Plane–Strain Problems

- The dimension of the structure in study in one direction, for example the $z$ coordinate direction, is very large in comparison with the dimension of the structure in the other two directions ($x$ and $y$).

- The applied forces act in the $x,y$ plane and do not vary in the $z$ direction.

- Important practical applications of this representation occur in the analysis of dams, tunnels and other geotechnical works.

- Also small-scale problems as bars and rollers compressed by forces normal to their cross section.
Plane–Strain Problems

- Practical examples of application:
Finite–Element Formulation specific to Plane flow

- Basic variational form for finite–element discretization:

\[ \delta \pi = \int_V \bar{\sigma} \, \delta \bar{\varepsilon} \, dV + \int_V \lambda \, \delta \varepsilon_v \, dV + \int_V \dot{\varepsilon}_v \, \delta \lambda \, dV - \int_{S_f} F_i \, \delta u_i \, dS = 0 \]

- for the Lagrange multiplier method or:

\[ \delta \pi = \int_V \bar{\sigma} \, \delta \bar{\varepsilon} \, dV + \int_V K \varepsilon_v \, \delta \varepsilon_v \, dV - \int_{S_f} F_i \, \delta u_i \, dS = 0 \]

- for the penalty function constrains.

\( \lambda \) is the Lagrange multiplier, \( K \) is a large positive constant, \( \varepsilon_v \) is the volumetric strain–rate and \( \dot{\varepsilon} \) is the effective strain–rate.
The effective strain–rate $\hat{\varepsilon}$ includes only non–zero components, namely:

$\dot{\varepsilon}_x$, $\dot{\varepsilon}_y$, and $\dot{\gamma}_{xy}$

A quadrilateral linear and isoparametric element is used for discretization.

The velocity field $\mathbf{u}$ is approximated by shape functions in terms of nodal point velocity values as:

$$\mathbf{u} = \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \mathbf{N}^T \mathbf{v}$$
It is convenient to arrange the strain–rate components in a vector form. For 2D elements, particularly plane–strain cases, the strain–rate components can be written as:

\[
\mathbf{\dot{\varepsilon}} = \begin{bmatrix}
\dot{e}_x \\
\dot{e}_y \\
\dot{e}_z \\
\dot{\gamma}_{xy}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial u_x}{\partial x} \\
\frac{\partial u_y}{\partial y} \\
0 \\
\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}
\end{bmatrix}
\]
The strain–rate vector can be written in the matrix form, called the strain–rate matrix (\( \mathbf{B} \)):

\[
\dot{\mathbf{e}} = \begin{bmatrix} \dot{e}_x \\ \dot{e}_y \\ \dot{e}_z \\ \dot{g}_{xy} \end{bmatrix} = \mathbf{Bv}
\]

where,

\[
\mathbf{B} = \begin{bmatrix}
X_1 & 0 & X_2 & 0 & X_3 & 0 & X_4 & 0 \\
0 & Y_1 & 0 & Y_2 & 0 & Y_3 & 0 & Y_4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Y_1 & X_1 & Y_2 & X_2 & Y_3 & X_3 & Y_4 & X_4
\end{bmatrix}
\]

In order to do this transformation, the strain–rate vectors must be represented in a unified form, where \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \), Represent \( \mathbf{u}_x \) and \( \mathbf{u}_y \) respectively.

The volumetric strain–rate \( \dot{\mathbf{e}}_v \) is expressed by:

\[
\dot{\mathbf{e}}_v = \dot{e}_x + \dot{e}_y + \dot{e}_z = \mathbf{C}^T \mathbf{v}
\]

\[
\mathbf{C}^T = \{1, 1, 1, 0\} \mathbf{B}
\]
The effective strain–rate $\hat{\varepsilon}$, in a discrete form, is defined by:

$$\hat{\varepsilon} = (\varepsilon^T D \varepsilon)^{1/2} = (v^T B^T D B v)^{1/2} = (v^T P v)^{1/2},$$
where $P = B^T D B$.

Matrix $D$ is given by:

$$D = \begin{bmatrix}
\frac{2}{3} & 0 \\
0 & \frac{1}{3}
\end{bmatrix}$$

The elements in the diagonal of matrix $D$, $2/3$ and $1/3$ correspond to normal strain–rate and engineering shear strain–rate, respectively.
Now, the basic equation, either for the Lagrange multiplier method or for the penalty function constrains, is discretized and a set of nonlinear simultaneous equations, called the stiffness equations, is obtained.

For the penalty function constrains, from the arbitrariness of $\delta v$, the stiffness equations obtained are:

$$\frac{\partial \pi}{\partial v} = \sum_i \left[ \int_{V_i} \frac{\bar{\sigma}}{\bar{\varepsilon}} P v \, dV + \int_{V_i} K C C^T v \, dV - \int_{s_f} N F \, dS \right] = 0$$

To obtain the solution, this equation is linearized using the Newton–Raphson method.
Example of Application: Closed–Die Forging with Flash

- Important operation in shaping metals into useful objects;

- The process involves the compression of material usually between two dies.

- The metal flow is restricted to fill the closed die cavity, and the excessive material flows through the gap between the closing dies and a flash is formed.

- The flash due to this excessive material is subsequently trimmed from the forging.
Schematic diagram of a plane-strain closed-die forging:

Lyapunov and Kobayashi conducted experiments to examine the metal flow in plane-strain closed-die forgings;

The results obtained from this experiment, including grid patterns, velocity fields, load-displacement curves, flash dimensions, and the height variations of the specimen, were compared with the results of rigid-plastic finite-element analysis.
The specimen used is made of pure lead – has a perfectly rigid plastic material behavior. Constant flow stress assumed to be 2500 psi. Dimensions were 50.8mm in height and width.

Two extreme cases of friction were considered:

- Perfect lubrication at the interfaces of die and workpiece = no frictional forces exist;
- Sticking condition, which means, when material touches the die, the surface of material is completely adhered to the die;

The initial mesh consists of 208 elements interconnected at 238 nodal points.

The nonsteady–state forging process was analyzed in a step–by–step manner with a die displacement at each step that was 1% of the initial height of the specimen.
Additional elements were provided at the stage of flash formation.

Results for sticking friction:

- Velocity distributions. On the left, the FEM, and on the right, the experiment.

- Material particles near the upper fillet move sideways in the early stages (1) and change direction downward in subsequent stages (2);

- The particles in the core portion remain stationary until the flash is formed and then begin to move upwards when the lower die cavity is almost filled (3).

- It is seen that the computed velocity distributions are in good agreement with the experimental observations.
Flash formation.

- Comparison of grid distortions between experiment (left) and theory, for both cases, sticking and frictionless (right). (Dark portions indicate rigid elements)

- Agreement between theory and experiment is good. However the study demonstrated that there is need for handling of complex die geometry more efficiently and establishment of a remeshing scheme for simulations of severe deformation, such as flash formation.
References
