Extended Finite Element Method XFEM

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Plan of presentation

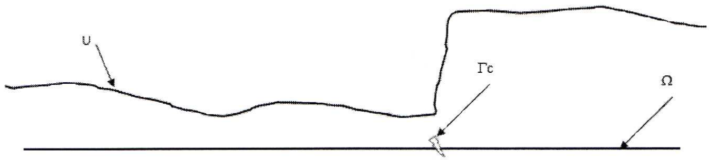
- Motivation
- General idea of XFEM
  - XFEM enrichment in 1D
  - XFEM enrichment in 2D
- Numerical Integration in XFEM
- Blending elements
- XFEM implementation
- Simple example

Motivation

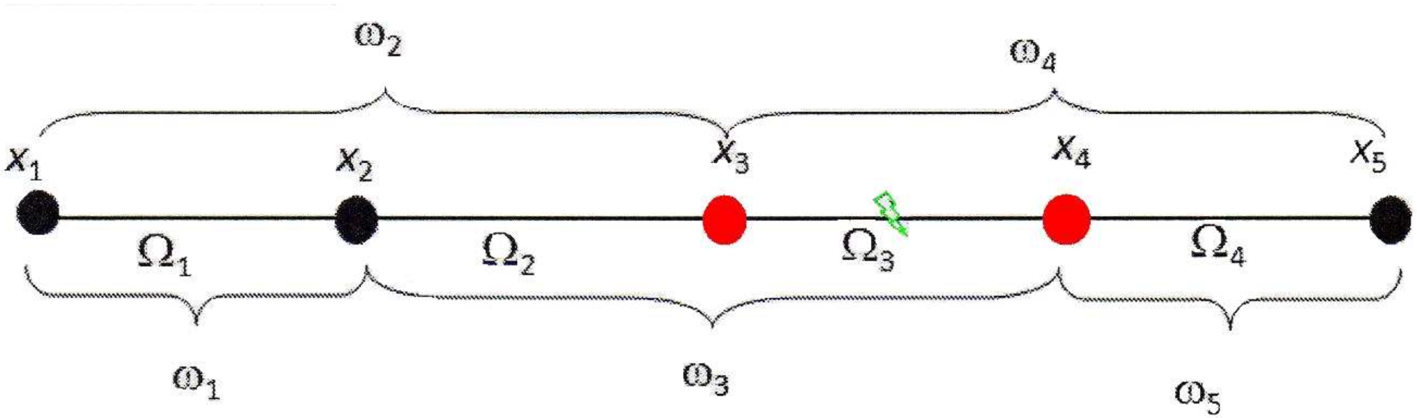
- Difficulties in modeling discontinuous field by FEM
- Necessity of remeshing in FEM
- High computational cost of FEM
- Low accuracy of FEM in modeling cracks

General idea of XFEM

- Discontinuous field U:

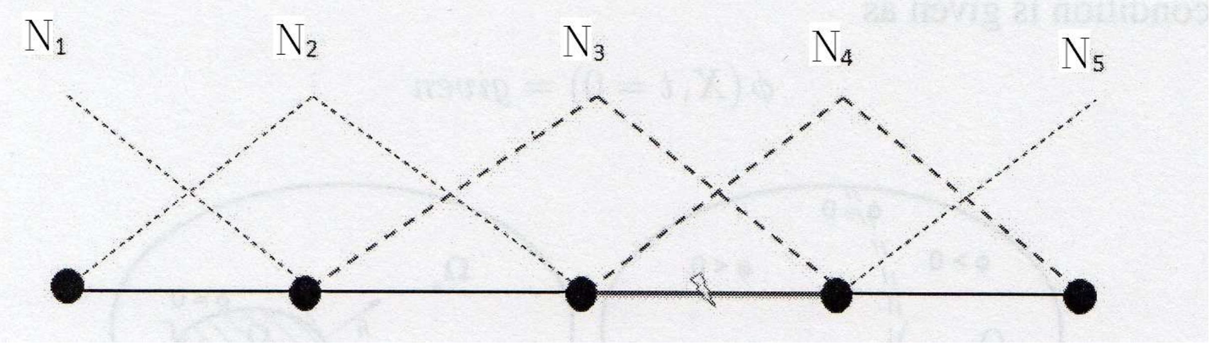


- XFEM mesh discretization and enriched nodes:



General idea of XFEM

- Standard FEM linear interpolation functions over 1D domain:

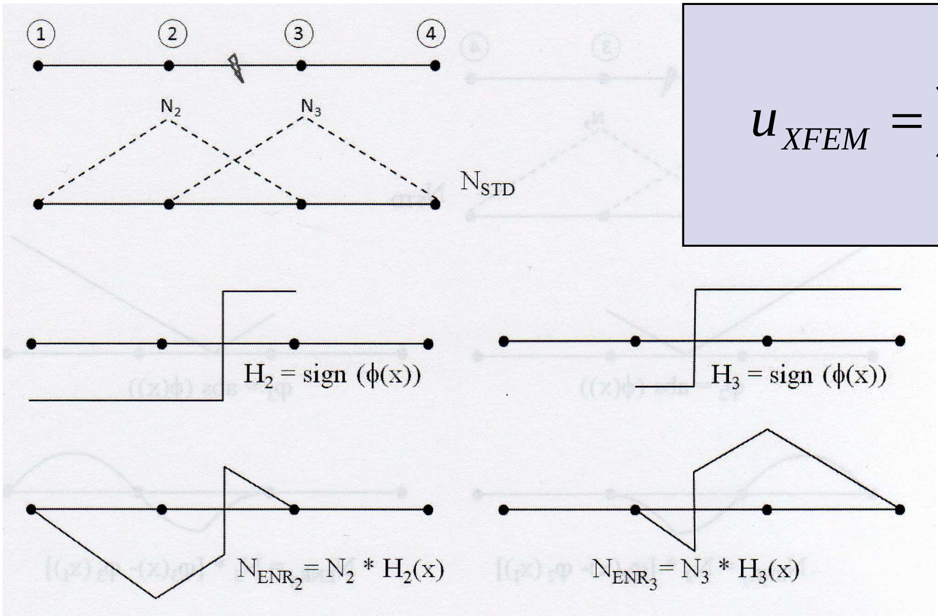


- Standard FEM approximation:

$$u^h(x) = \sum_{\forall l} N_l(x) u_l$$

General idea of XFEM

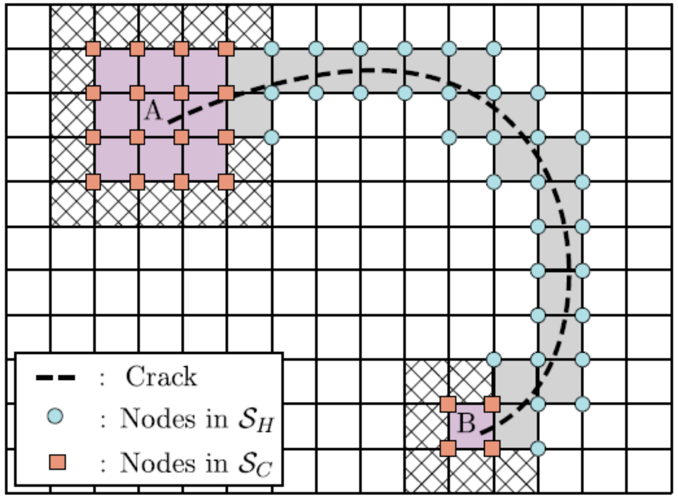
- Enriched basis function for a strong discontinuity in 1D



$$u_{XFEM} = \sum_{i=1}^{N} N_i u_i + \sum_{j=1}^{n^E} N_j H(x) a_j$$

General idea of XFEM

- XFEM mesh in 2D – enriched nodes

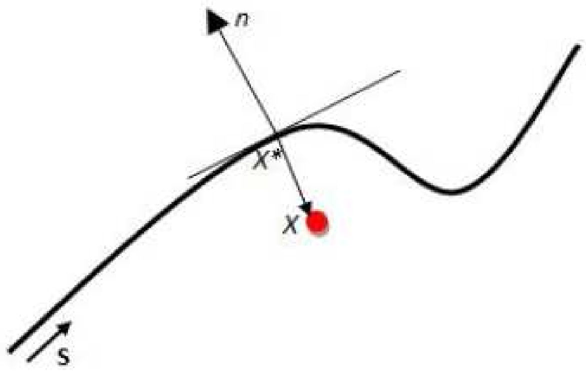


- In this case 2 types of enrichment is used:
  - Heaviside/step function in elements cut by crack
  - Asymptotic near-tip enrichment function for the elements which contain crack tip

General idea of XFEM

- XFEM approximation in 2D domain with crack tip:

$$H(x,y)=\begin{cases}1 & for(x-x^*).n>0\\ -1 & for(x-x^*).n<0\end{cases}$$

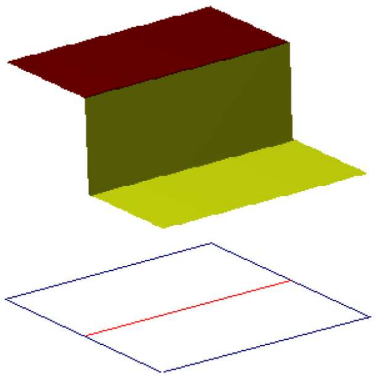


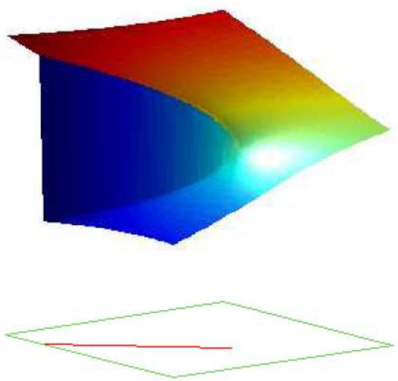
$$\{F_l(r,\theta)\}_{l=1}^{4}=\left\{\sqrt{r}\cos\left(\frac{\theta}{2}\right),\sqrt{r}\sin\left(\frac{\theta}{2}\right),\sqrt{r}\sin\left(\frac{\theta}{2}\right)\sin\theta,\sqrt{r}\cos\left(\frac{\theta}{2}\right)\sin\theta\right\}$$

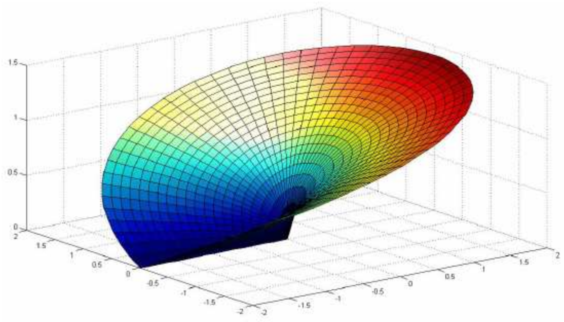
$$u^h(x)=\sum_{\forall l}N_l(x)u_l+\sum_{j\in S_H}N_j(x)H(x)q_j^0+\sum_{j}\sum_{k\in S_C}N_k(x)F_j(x)q_k^j$$

General idea of XFEM

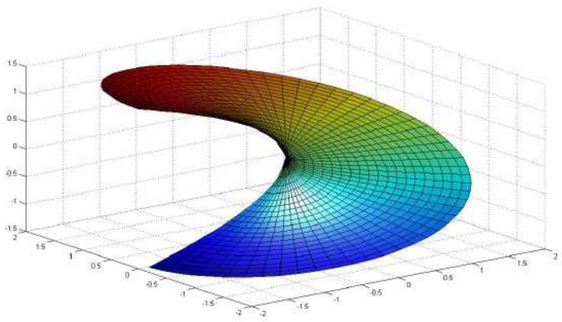
- Enriched basis for crack growth in 2D:



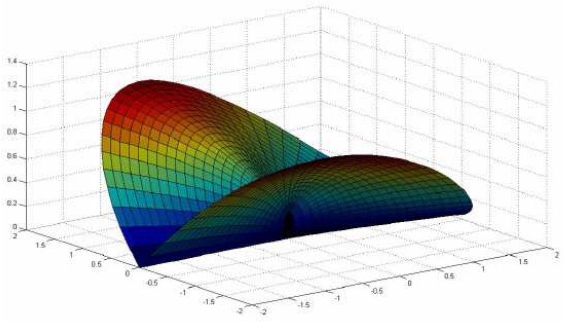




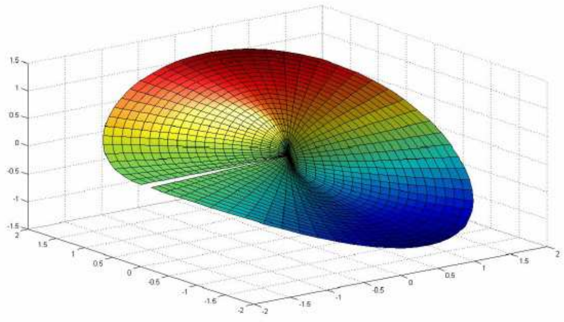
(a) $\sqrt{r}\ cos\left(\frac{\theta}{2}\right)$



(b) $\sqrt{r}\ sin\left(\frac{\theta}{2}\right)$



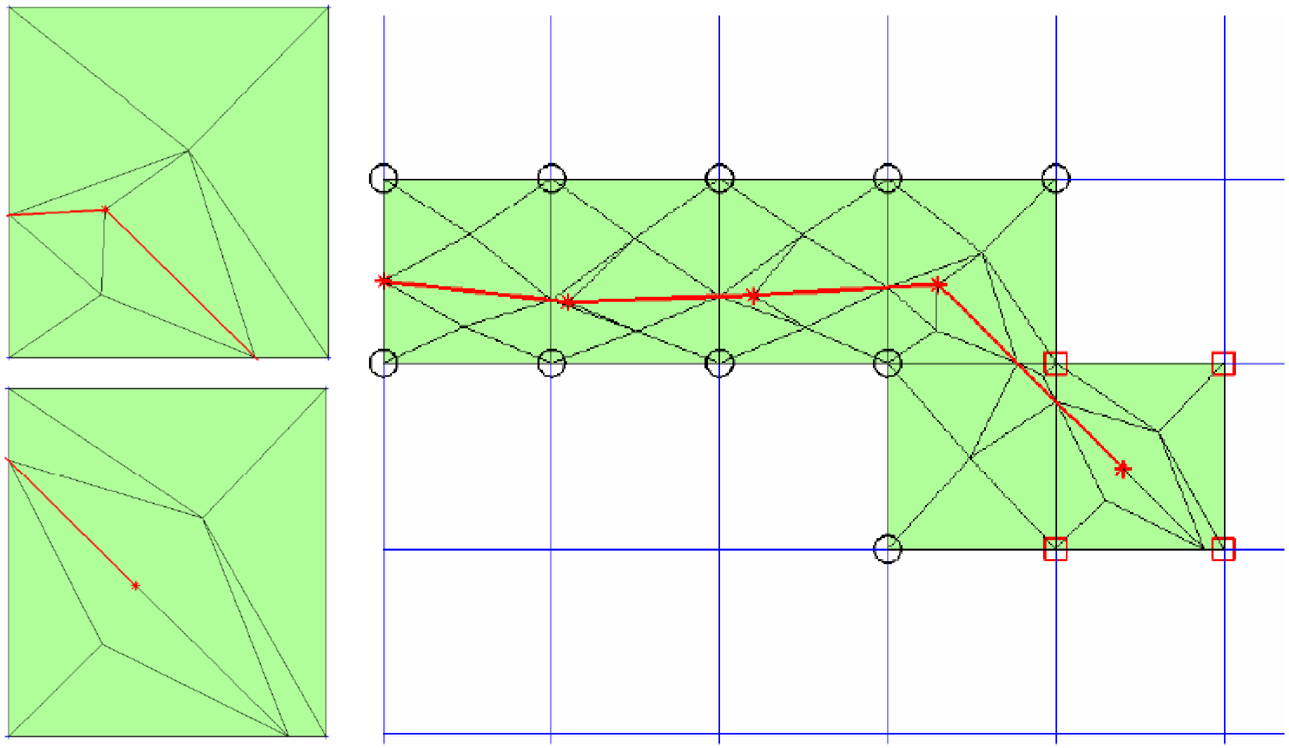
(c) $\sqrt{r}\ sin\left(\frac{\theta}{2}\right)\ sin\theta$



(d) $\sqrt{r}\ cos\left(\frac{\theta}{2}\right)\ sin\theta$

Numerical Integration in XFEM

- For elements cut by the crack, modified integration scheme is practiced in XFEM



Numerical Integration in XFEM

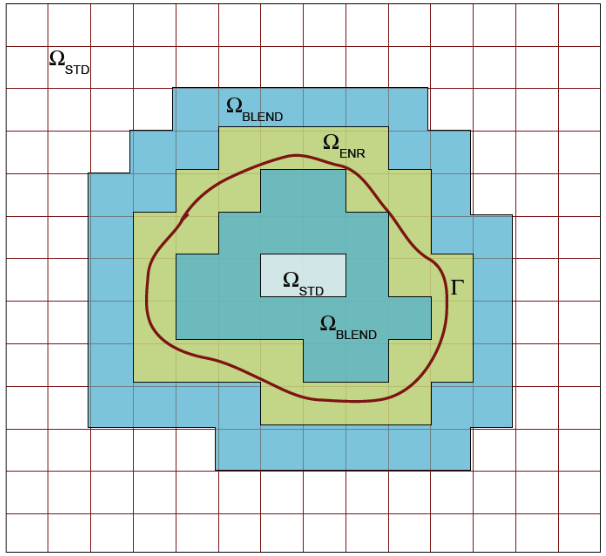
- The partitioning of an element is done only for integration purpose and no extra degrees of freedoms are added to the system unlike the usual FEM

- No conditions on the shape of sub-polygons or sub-triangles is imposed

Blending elements

- Domain Ω_{BLEND} consist elements whose some of the nodes are enriched and some of them are not. As a results of this the enrichment function is not reproduced exactly in blending elements.

- Shifted enrichment automatically removes the enrichment from the domain which is not required to be enriched:



$$u^h(x) = \underbrace{\sum_{\forall I} N_I(x) u_I}_{\text{Standard FEM}} + \underbrace{\sum_{J \in S_H} N_J(x)\left[H(x) - H(x_J)\right] a_J^0}_{\text{Step enrichment function}} + \underbrace{\sum_j \sum_{K \in S_C} N_K(x)\left[F^{(j)}(x) - F^{(j)}(x_K)\right] a_K^{(j)}}_{\text{Near-tip enrichment function}}$$

Crack initiation and growth

- Some of the commonly used crack growth criteria are:
  - Minimum strain energy density criteria
  - Maximum energy release rate criteria
  - Maximum hoop stress or maximum principal stress criteria
  - Global tracking algorithm

Crack initiation and growth

- Minimum strain energy density criteria

  - The crack initiation will occur when the minimum of the strain energy density function S reaches to some critical value S_{cr}

  - The crack will extend in a direction in which strain energy density factor possess a minimum value

Crack initiation and growth

- The minimum strain energy density factor S is given by:

$$S = a_{11}K_I^2 + 2a_{12}K_I K_{II} + a_{22}K_{II}^2 + a_{33}K_{III}^2$$

$$a_{11} = \frac{\kappa+1}{16\mu\lambda\kappa^2\cos\theta}\left[2(1-2\nu)+\frac{\kappa-1}{\kappa}\right]$$

$$a_{12} = \frac{(\kappa^2-1)^{1/2}}{8\mu\lambda\kappa^2\cos\theta}\left[\frac{1}{\kappa}-(1-2\nu)\right]$$

$$a_{22} = \frac{1}{16\mu\lambda\kappa^2\cos\theta}\left[4(1-\nu)(\kappa-1)+\frac{1}{\kappa}(\kappa+1)(3-\kappa)\right]$$

$$a_{33} = \frac{1}{4\mu\lambda\kappa\cos\theta}$$

- where K_I, $K_{II,}$ K_{III} are the mode I, II, III stress intensity factors.

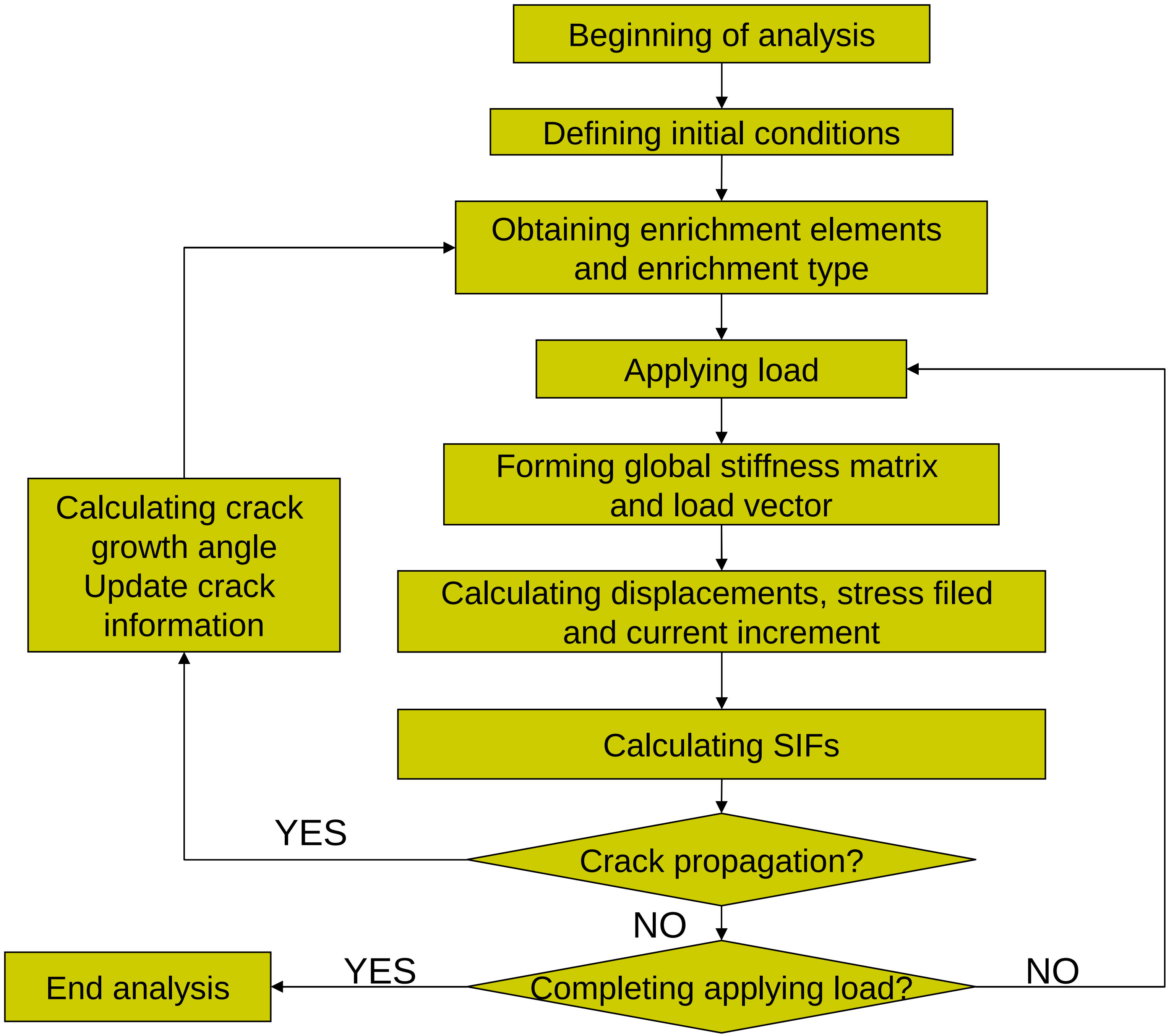
Crack initiation and growth

- The direction of propagation is determined such that:

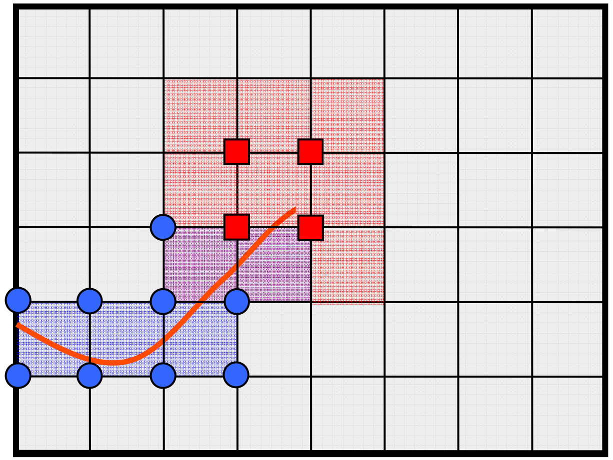
$$\left(\frac{\partial S}{\partial \theta}\right)_{\theta=\theta_{cr}} = 0$$

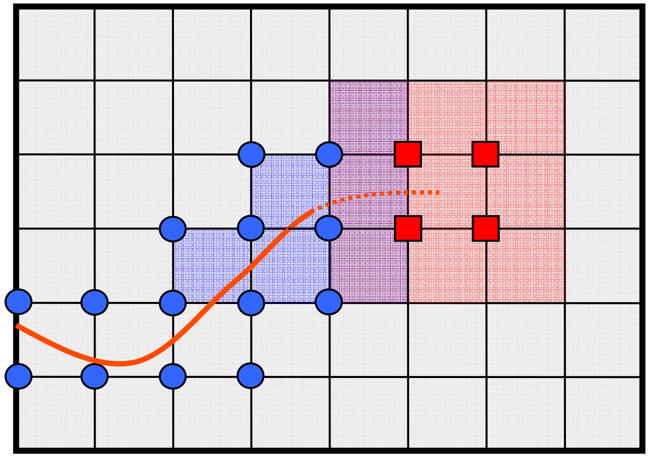
$$\left(\frac{\partial^2 S}{\partial^2 \theta}\right)_{\theta=\theta_{cr}} > 0$$

- It is worth mentioning that the criteria works well for linear elastic fracture mechanics


Beginning of analysis
Defining initial conditions
Obtaining enrichment elements and enrichment type
Applying load
Forming global stiffness matrix and load vector
Calculating displacements, stress filed and current increment
Calculating SIFs
Crack propagation?
YES
NO
Calculating crack growth angle Update crack information
Completing applying load?
YES
NO
End analysis

Updating stiffness matrix K

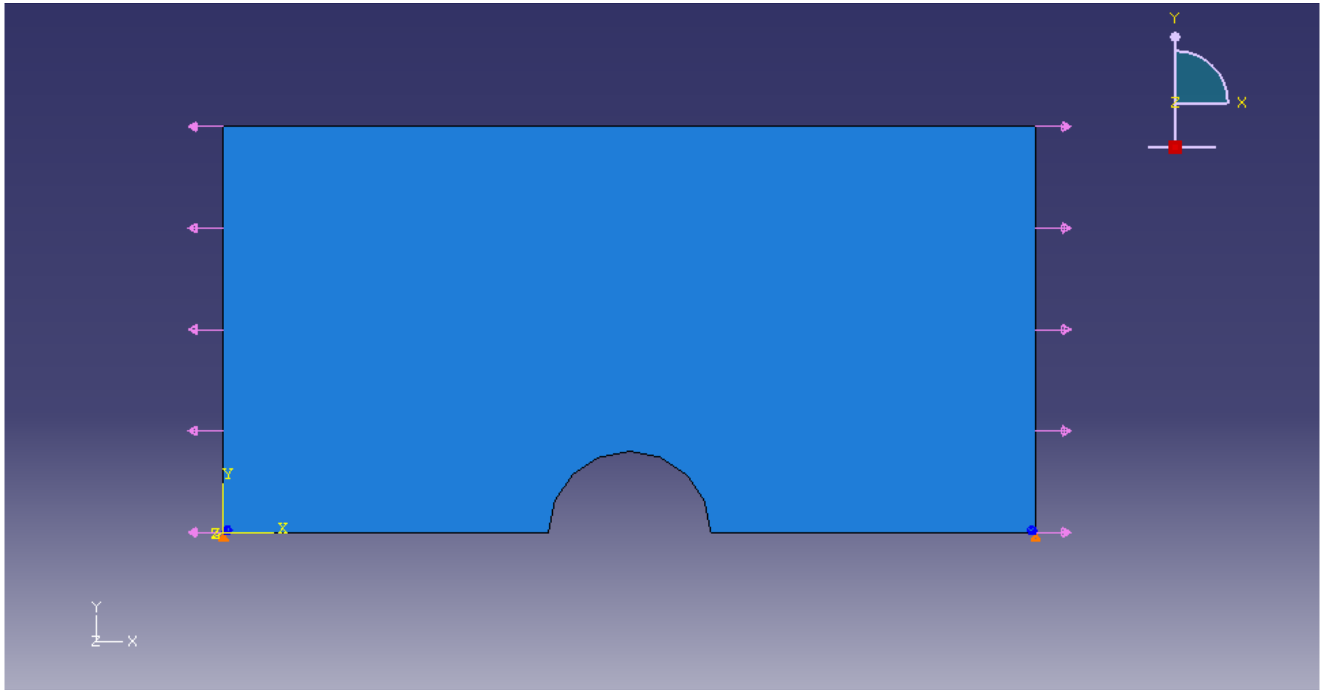




$$\begin{bmatrix} K_{uu}^{1} & K_{ua}^{1} & K_{ub}^{1} \\ K_{ua}^{1\,T} & K_{aa}^{1} & K_{ab}^{1} \\ K_{ub}^{1\,T} & K_{ab}^{1\,T} & K_{bb}^{1} \end{bmatrix} \xrightarrow{\text{Constant}} \begin{bmatrix} K_{uu}^{1} & K_{ua}^{1} & K_{ua}^{2} & K_{ub}^{2} \\ K_{ua}^{1\,T} & K_{aa}^{1} & 0 & 0 \\ K_{ua}^{2\,T} & 0 & K_{aa}^{2} & K_{ab}^{2} \\ K_{ub}^{T} & 0 & K_{ab}^{2\,T} & K_{bb}^{2} \end{bmatrix}$$

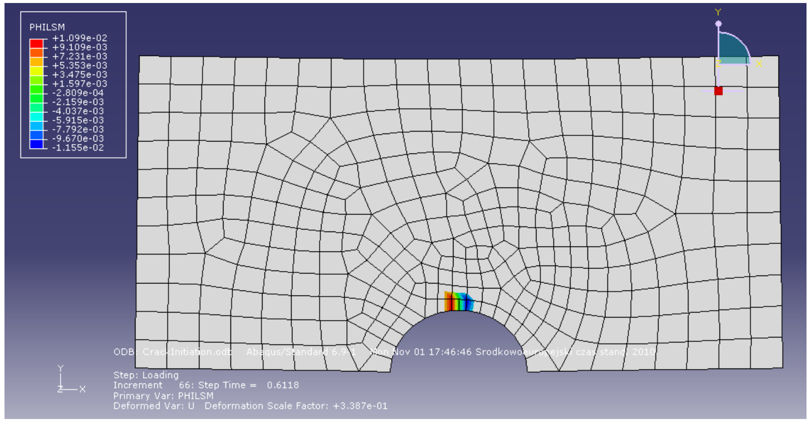
Example

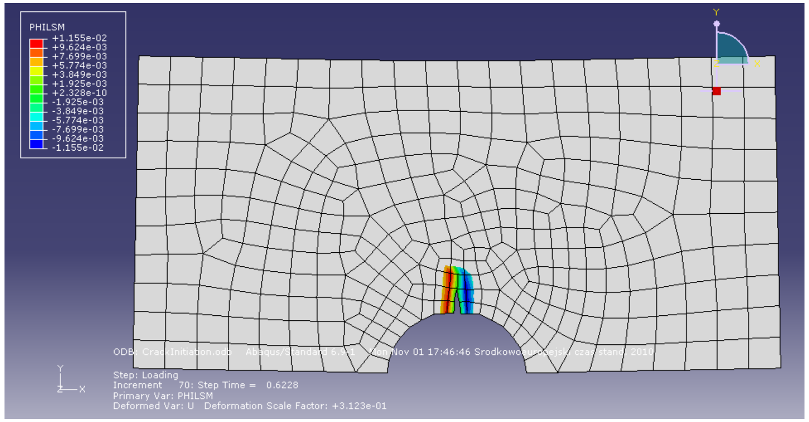
- Initiation of crack in 2D Plate – ABAQUS 6-9

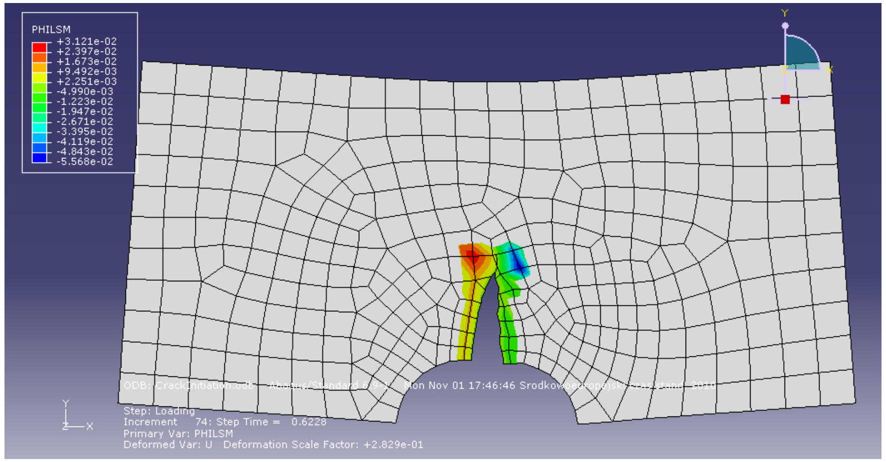


Example

- Results: Enrichment functions values

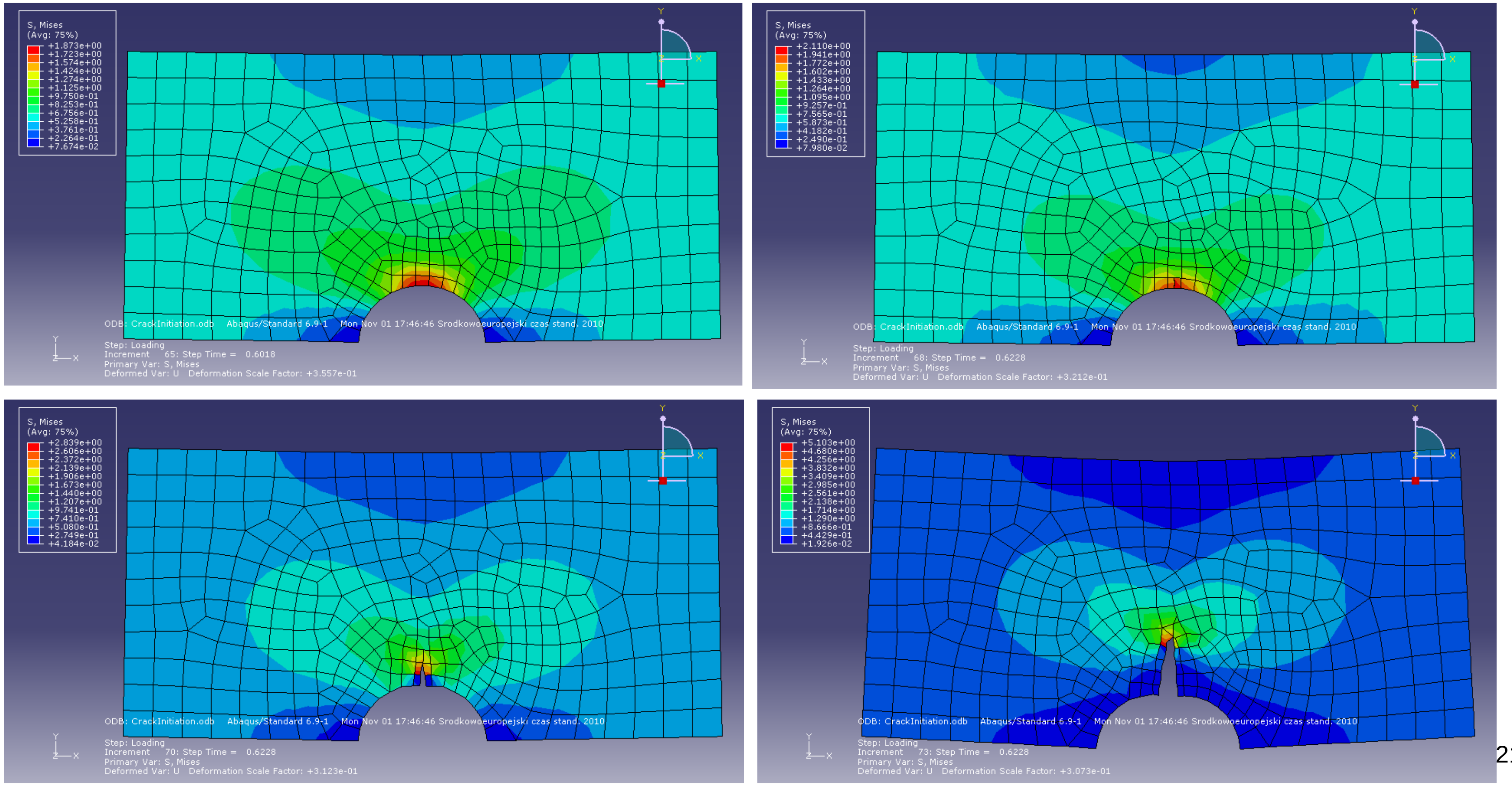






Example

- Results: Stress relaxation



Conclusion

- Better accuracy of XFEM in comparison to FEM
- Re-meshing is not need
- Possibility of modeling discontinuities and singularities by XFEM
- Lower computationally cost

References

- T. Belytschko, R. Gracie, G. Ventura, *A review of extended/generalized finite element methods for material modeling*, Modeling Simul. Mater. Sci. Eng. 17 (2009)

- Y. Abdelaziz, A. Hamouine, *A survey of the extended finite element*, Computers & Structures, Volume 86, Issue 11-12, June 2008, Pages 1141-1151

- N. Moës, J. Dolbow, T. Belytschko, *A finite element method for crack growth without remeshing*, Int. J. Numer. Meth. Engng. 46, 131-150 (1999)

- http://www.xfem.rwth-aachen.de/

- http://dilbert.engr.ucdavis.edu/~suku/xfem/

Thank you for your attention